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AN ANALYSIS OF RATE- AND MATERIAL PARAMETER-DEPENDENT LIMITING CASES IN VISCOPLASTICITY LAWS

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Abstract With reference to a well known viscoplasticity law, it is shown how the equilibrium curve associated with a process can be determined. Furthermore, for this viscoplasticity law, an analysis is given for the rate- and material parameter-dependent limit response. In particular, it turns out that for very slow motions, the strain–stress relation converges everywhere to a plasticity model. However, the time derivatives of the variables in this plasticity law are not everywhere equal to the limit for very slow motions of the corresponding time derivatives in the viscoplasticity law.

1. INTRODUCTION

An analysis of the limit response for very fast and very slow motions in the case of the general linear viscoelasticity law is given by Gurtin and Herrera (1965). In the case of nonlinear viscoelasticity laws as well as viscoplasticity laws, given in the one-dimensional form

$$\dot{\sigma} = E(\varepsilon, \sigma)\dot{\varepsilon} + G(\varepsilon, \sigma),$$

the corresponding limit response is discussed by Gurtin *et al.* (1980). Here, σ and ε are the one-dimensional stress and strain, respectively, $\dot{\sigma}$ and $\dot{\varepsilon}$ the corresponding material time derivatives, and E denotes a smooth and strictly positive function. The function G was postulated to be smooth for viscoelastic materials, and piece-wise smooth with $G \equiv 0$ on a suitable region for viscoplastic materials. Note that the analysis of Gurtin and Herrera (1965), as well as of Gurtin *et al.* (1980), is based on the idea of accelerating and retarding a given strain history.

The present work is concerned with viscoplasticity laws represented in three-dimensional form, and formulated by means of internal variables. The characteristic feature of the evolution equations for the internal variables used is that they fall in the general framework for defining viscoplasticity laws postulated by Kratochvill and Dillon (1969). For computational simplicity, only kinematic and isotropic hardening are considered here. In particular, we consider a well known viscoplasticity law in which nonlinear kinematic hardening properties of the Armstrong-Frederick type are incorporated [see Armstrong and Frederick (1966) as well as Chaboche (1977)]. The purpose of this work is to calculate for this viscoplasticity law the equilibrium curve associated with a process, and to investigate the limiting cases for very fast and very slow motions. In addition to the limit response of the strain–stress relation, the limit response of the time derivatives of the variables involved in the system of constitutive equations is also discussed. To that end, we follow the above works of Gurtin *et al.* by using the idea of accelerating and retarding a given strain history. As it will prove, this analysis is equivalent to considering corresponding limit processes depending on the viscosity parameter in the constitutive relations. Note that no attempt is made to comment on the physical relevance of the viscoplasticity model and its limit responses.

After introducing some notations and definitions in Section 2, the viscoplasticity law to be investigated is given in Section 3. The structure of this viscoplasticity law is based on the concept of so-called overstresses, and it is shown how the equilibrium curve associated with a process is to be established. In Section 4, various limits for very fast and very slow motions are calculated. In particular, we obtain for very slow motions the result that the system of constitutive equations reduces to a plasticity law.† However, it is shown in Section 5 that the time derivatives of the variables in this plasticity law are not everywhere equal to the limit for very slow motions of the corresponding time derivatives in the viscoplasticity law. Note that some of the results derived in Sections 4 and 5 are given by Haupt *et al.* (1992) as well, but from the technical point of view, the proof given here differs from that given by Haupt *et al.* In addition, Section 5 deals with some time transformations which can be applied to extend the viscoplasticity law considered in order to obtain a nonlinear dependence on the overstress used. Finally, the analysis for the rate-dependent limiting cases on the basis of material parameters is briefly sketched.

2. PRELIMINARIES

Let R be the real axis. For $a \in R$, $|a|$ is the absolute value of a. We write t for the time variable, and denote by $\dot{f}(t)$ the material time derivative of the function $f(t)$, where $t \in I$ and I is an interval of R. We say that the function $f(t)$ is smooth if $\dot{f}(t)$ exists at each $t \in I$, and if the function $\dot{f}(t)$ is continuous on I. For $x \in R$, $\langle x \rangle$ represents the function

$$\langle x \rangle := \begin{cases} x & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Since the formulation is not affected by a space dependence, an explicit reference to space will be dropped. In this work, so-called static recovery-effects are not taken into account. Furthermore, the material parameters used take values on the real interval $(0, \infty)$, and the deformations considered are small isothermal deformations.

We use bold-face letters for second-order tensors and write $\mathscr{L}$ for the set of all second-order tensors. In particular, $\mathbf{1}$ represents the identity second-order tensor, and $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of $\mathbf{A}$. We write tr $\mathbf{A}$ for the trace of $\mathbf{A}$, $\mathbf{A}^{\mathrm{D}} = \mathbf{A} - \frac{1}{3}(\mathrm{tr}\,\mathbf{A})\mathbf{1}$ for the deviator of $\mathbf{A}$, $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}\,(\mathbf{A}\mathbf{B}^{\mathrm{T}})$ for the inner product of $\mathbf{A}$ and $\mathbf{B}$, as well as $\|\mathbf{A}\| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$ for the euclidean norm of $\mathbf{A}$. We also use for $\mathbf{A}^{\mathrm{D}}$ the norm $\|\mathbf{A}^{\mathrm{D}}\|_{\mathrm{D}} = \sqrt{\frac{2}{3}\mathbf{A}^{\mathrm{D}} \cdot \mathbf{A}^{\mathrm{D}}}$, referred to as the deviatoric norm of $\mathbf{A}^{\mathrm{D}}$.

Let $\mathbf{E} : I \to \mathscr{L}$ be a strain curve, i.e. a function that assigns each time t in I a linearized Green strain tensor $\mathbf{E}(t) \in \mathscr{L}$. Analogously, a stress curve is a function $\mathbf{T} : I \to \mathscr{L}$, i.e. a function that assigns each time t in I a Cauchy stress tensor $\mathbf{T}(t) \in \mathscr{L}$. Generally, it is assumed that all derivatives up to any desired order exist. If I is half-open or closed, the derivatives on the boundary points are to be understood as one-sided. It is common to denote a continuous strain curve (stress curve) on $(-\infty, \infty)$ as a strain history $\mathbf{E}(\cdot)$ (stress history $\mathbf{T}(\cdot)$). All the time functions to be regarded below can be considered as defined in the interval $[0, T]$, $0 < T < \infty$, if an appropriate translation in the time is applied. Hence, without loss of generality, we set in the following $I = [0, T]$. In particular, $\mathbf{E}(t = 0) \neq \mathbf{0}$ is not excluded.

In this paper, we are concerned with constitutive functionals which are represented implicitly, e.g. by a system of ordinary differential equations for a certain set of variables (state variables) which is denoted by $\mathscr{S}$ [see also Haupt (1992) in this context]. Generally, the notion of a process with duration T, e.g. induced by a given strain curve $\mathbf{E} : [0, T] \to \mathscr{L}$, is understood as follows. We suppose that the state of the considered material system is known for each $t \in [0, T]$ if $\mathscr{S}(t)$ is known. The map assigning to each $t \in [0, T]$ a value $\mathscr{S}(t)$ in the space of all state variables is called a strain-controlled process if the map is

† We do not attempt to give here a general definition on (rate-independent) plasticity laws. Such definitions are given, e.g., by Del Piero (1985), Lucchesi and Silhavy (1991), and Bertram (1992). In these references, use is made of the so-called concept of elastic ranges initially introduced by Pipkin and Rivlin (1965), and Owen (1970). For the purpose of our work it suffices to think of plasticity models as systems of constitutive relations having properties as those defined, e.g., by Hill (1967), Lubliner (1974), Dafalias and Popov (1976), and Casey and Naghdi (1984).

compatible with the postulated system of constitutive equations. A stress-controlled process is defined in exactly the same way if a stress curve on $[0, T]$ is given instead of a strain curve. For example, a relaxation process (creep process) is a process for which $\mathbf{E}(t) = \text{const.}$ ($\mathbf{T}(t) = \text{const.}$) holds. Thus, roughly speaking, $\mathscr{S}(t)$ consists of all the variables for which initial conditions are needed in order to calculate the material response after a time increment, if loading conditions of stress or strain are given. We denote the set $\{\mathscr{S}(t)/t \in I\}$ as an $\mathscr{S}$-path, or as an $\mathscr{S}$-trajectory, or simply a state trajectory. A point $\mathscr{S}^*$ is called an equilibrium point if it represents an equilibrium solution of the differential equations governing the material response.

It is assumed that every relaxation process converges for an infinitely long duration to a corresponding equilibrium state. This motivates the introduction of a so-called equilibrium curve as follows. Together with a strain curve $\mathbf{E}(\cdot)$, each time $t \in I$ is assigned an equilibrium state $\mathscr{S}^{(E)}(t)$. The point $\mathscr{S}^{(E)}(t)$ can be interpreted as that obtained in a relaxation process with infinite long duration. This relaxation process corresponds to that process where the strain is imagined to be held constant at the value $\mathbf{E}(t)$, and begins at the point $\mathscr{S}(t)$ belonging to the actual process induced, e.g. by the strain curve $\mathbf{E}(\cdot)$. Then the map assigning to each $t \in I$ a point $\mathscr{S}^{(E)}(t)$, so obtained, is called the equilibrium curve belonging to the process $\mathscr{S}(\cdot)$. Analogously, the set $\{\mathscr{S}^{(E)}(t)/t \in I\}$ is called an $\mathscr{S}^{(E)}$-path, or an $\mathscr{S}^{(E)}$-trajectory, or simply an equilibrium trajectory.

3. THE VISCOPLASTICITY LAW

Consider the following system of constitutive equations:

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}_e + \dot{\mathbf{E}}_i, \tag{1}$$

$$\mathbf{T} = 2\mu\mathbf{E}_e + \lambda(\operatorname{tr}\mathbf{E}_e)\mathbf{1}, \tag{2}$$

$$F(t) = \bar{F}(\mathbf{T}, \boldsymbol{\xi}, k) := \bar{f}(\mathbf{T}, \boldsymbol{\xi}) - k, \tag{3}$$

$$f(t) = \bar{f}(\mathbf{T}, \boldsymbol{\xi}) := \sqrt{\tfrac{3}{2}(\mathbf{T} - \boldsymbol{\xi})^{\mathrm{D}} \cdot (\mathbf{T} - \boldsymbol{\xi})^{\mathrm{D}}}, \tag{4}$$

$$\dot{\mathbf{E}}_i = \frac{\langle F \rangle}{r}\mathbf{R}_i, \tag{5}$$

$$\mathbf{R}_i := \frac{\partial \bar{F}(\mathbf{T}, \boldsymbol{\xi}, k)}{\partial \mathbf{T}} = \frac{3}{2f}(\mathbf{T} - \boldsymbol{\xi})^{\mathrm{D}}, \tag{6}$$

$$\dot{\boldsymbol{\xi}} = c\dot{\mathbf{E}}_i - b\dot{s}\boldsymbol{\xi}, \tag{7}$$

$$\dot{k} = \beta(\gamma - k)\dot{s}, \tag{8}$$

$$\dot{s} := \|\dot{\mathbf{E}}_i\|_{\mathrm{D}} = \frac{1}{r}\langle F \rangle. \tag{9}$$

$\mu, \lambda, r, c, b, \gamma$: material constants.

Here, $\mathbf{E}_i$ and $\mathbf{E}_e$ denote the elastic and inelastic parts, respectively, in the additive decomposition of $\mathbf{E}$, eqn (2) is referred to as the spontaneous elasticity law, $\boldsymbol{\xi}$ is the back stress, k represents the variable describing isotropic hardening, and s stands for the arc length of $\mathbf{E}_i$ with respect to the norm $\|\cdot\|_{\mathrm{D}}$.

The set of all stress tensors $\mathbf{T}$ satisfying $F = 0$ for fixed $\boldsymbol{\xi}$ and k describes a so-called *static yield surface* in the space of all stress tensors. Analogously, the set of all stress tensors $\mathbf{T}$ satisfying an equation of the form $F = \text{const.} > 0$ for fixed $\boldsymbol{\xi}$ and k, describes a so-called *dynamic yield surface* in the space of all stress tensors. The function F defines a so-called overstress with respect to the static yield surface.†

The characteristic feature of the viscoplasticity law‡ (1)–(9) is the evolution equation for kinematic hardening due to Armstrong and Frederick (1966). This viscoplasticity model is included in the many works by Chaboche, and is investigated by him and coauthors in several aspects [see e.g. Chaboche (1977, 1993), Chaboche and Rousselier (1983) and references cited therein]. We note in passing that the model (1)–(9) falls in the general framework for the definition of viscoplasticity laws developed by Kratochvill and Dillon (1969).

The variables

$$\mathbf{E}, \mathbf{E}_i, \boldsymbol{\xi}, k \tag{10}$$

form a set $\mathcal{S}$ of state variables, as defined in Section 2. For a given strain or stress curve and prescribed initial conditions for (10), the system of constitutive equations (1)–(9) implies a strain- or stress-controlled process, respectively. Alternatively (and equivalently) one can choose, e.g. $\mathbf{T}$ instead of $\mathbf{E}$, or s instead of k, as state variables.

3.1. *An equivalent representation for the overstress* F

For later reference, we mention here that the overstress F can be written as well in terms of the differential equation

$$\dot{F} + \frac{\Theta}{r}\langle F \rangle = 2\mu(\mathbf{R}_i \cdot \dot{\mathbf{E}}), \tag{11}$$

where

$$\Theta(t) = \bar{\Theta}(\mathbf{T}(t), \boldsymbol{\xi}(t), k(t)) := \tfrac{3}{2}(c+2\mu) - b(\boldsymbol{\xi} \cdot \mathbf{R}_i) + \beta(\gamma - k) \tag{12}$$

and

$$\Theta(t) > 3\mu > 0. \tag{13}$$

Equation (11) follows from eqns (1)–(9). To obtain inequality (13), we point out that, for the viscoplasticity law (1)–(9), the relations

$$k_0 \leqslant k < \gamma, \tag{14}$$

$$0 \leqslant \|\boldsymbol{\xi}\|_D < \frac{c}{b}, \tag{15}$$

$$0 \leqslant \|\mathbf{T}^D - \boldsymbol{\xi}\|_D = \tfrac{2}{3} f. \tag{16}$$

are satisfied in any process, where initial conditions

$$\boldsymbol{\xi}(s=0) = \mathbf{0} \quad \text{and} \quad k(s=0) = k_0 < \gamma \quad \text{for} \quad \mathbf{E}_i(s=0) = \mathbf{0}$$

are given. Note in passing that this constitutive model cannot describe cyclic softening in view of expression (14). From the Cauchy–Schwarz inequality in the form

† For a discussion of such notions as well as of rate-dependent constitutive equations formulated by means of overstresses, see also Tsakmakis (1994).

‡ On the notion 'viscoplasticity law' see also Section 5 of the present work.

$$\tfrac{2}{3}|\xi\cdot(\mathbf{T}-\xi)^{\mathrm{D}}| \leqslant \|\xi\|_{\mathrm{D}}\,\|\mathbf{T}^{\mathrm{D}}-\xi\|_{\mathrm{D}},$$

as well as eqn (16), we have

$$\xi\cdot(\mathbf{T}-\xi)^{\mathrm{D}} \leqslant |\xi\cdot(\mathbf{T}-\xi)^{\mathrm{D}}| \leqslant f\,\|\xi\|_{\mathrm{D}}.$$

In view of expression (15), these inequalities imply

$$\frac{3c}{2}-\frac{3b}{2f}\,\xi\cdot(\mathbf{T}-\xi)^{\mathrm{D}} > 0.$$

The result (13) follows from this inequality, the relation (14), as well as the definition (12) for Θ.

To solve the differential eqn (11), the initial condition needed is given by

$$F(t=0) = F(0) = \bar{F}(\mathbf{T}(0), \xi(0), k(0)),$$

where

$$\mathbf{T}(0) := \mathbf{T}(t=0),$$

$$\xi(0) := \xi(t=0),$$

$$k(0) := k(t=0).$$

If $F < 0$, the trivial case of pure elasticity applies. If inelastic loading is involved, i.e. if

$$\langle F(t)\rangle = \begin{cases} F(0) \geqslant 0 & \text{if} \quad t=0 \\ F(t) > 0 & \text{if} \quad t\in(0,T] \end{cases}. \tag{17}$$

holds, then the differential equation

$$\dot{F}+\frac{\Theta}{r}F = 2\mu(\mathbf{R}\cdot\dot{\mathbf{E}}), \tag{18a}$$

$$F(t=0) = F(0) \tag{18b}$$

for F applies, which follows from eqn (11).

A differential equation corresponding to eqn (18) was first utilized by Kratochvill and Dillon (1969) in order to discuss limits in viscoplasticity laws depending on material parameters. The differential equation (18) plays as well an important part in the discussion of the rate-dependent limits in the constitutive model (1)–(9), to be given in what follows. Note that eqn (18) is the starting point in the analysis given by Haupt *et al.* (1992). Not all the results obtained in the present work, however, are obtained in Haupt *et al.* In addition, as was mentioned in the Introduction, the mathematical proof of the various results given here is different from that provided by Haupt *et al.*

3.2. *The associated equilibrium curve*

For a given strain- or stress-controlled process, the strain-stress relation governing the associated equilibrium curve is given by

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}_{e}^{el}+\dot{\mathbf{E}}^{in}, \tag{19}$$

$$F = \bar{F}(\mathbf{T}, \xi, k) < 0; \tag{20}$$

$$\dot{\mathbf{T}}^{(e)} = 2\mu\dot{\mathbf{E}}+\lambda(\mathrm{tr}\,\dot{\mathbf{E}})\mathbf{1}. \tag{21}$$

$$\mathbf{E}_i^{(E)} = \mathbf{E}_i = \text{const.}, \quad \xi^{(E)} = \xi = \text{const.}, \quad k^{(E)} = k = \text{const.} \tag{22}$$

$$\underline{F = \bar{F}(\mathbf{T}, \xi, k) \geqslant 0:} \tag{23}$$

$$\dot{\mathbf{T}}^{(E)} = 2\mu\dot{\mathbf{E}}_e^{(E)} + \lambda(\operatorname{tr}\dot{\mathbf{E}})\mathbf{1}, \tag{24}$$

$$F^{(E)}(t) := \bar{F}(\mathbf{T}^{(E)}(t), \xi^{(E)}(t), k^{(E)}(t)), \tag{25}$$

$$\dot{\mathbf{E}}_i^{(E)} = \dot{l}^{(E)}\mathbf{R}_i^{(E)}, \tag{26}$$

$$\mathbf{R}_i^{(E)} := \frac{\partial \bar{F}(\mathbf{T}^{(E)}, \xi^{(E)}, k^{(E)})}{\partial \mathbf{T}^{(E)}} = \frac{3}{2k^{(E)}}(\mathbf{T}^{(E)} - \xi^{(E)})^D, \tag{27}$$

$$\dot{\xi}^{(E)} = c\dot{\mathbf{E}}_i^{(E)} - b\dot{l}^{(E)}\xi^{(E)}, \tag{28}$$

$$\dot{k}^{(E)} = \beta(\gamma - k^{(E)})\dot{l}^{(E)}, \tag{29}$$

$$\dot{F}^{(E)} = 0 \Rightarrow \dot{l}^{(E)} = 2\mu\frac{\mathbf{R}_i^{(E)} \cdot \dot{\mathbf{E}}}{\Theta^{(E)}}, \tag{30}$$

$$\Theta^{(E)} := \tfrac{3}{2}(c + 2\mu) - b(\xi^{(E)} \cdot \mathbf{R}_i^{(E)}) + \beta(\gamma - k^{(E)}). \tag{31}$$

For clarity, the variables $\mathbf{E}, \mathbf{T}, \mathbf{E}_i, s, F, \xi, k$ are replaced by the variables $\mathbf{E}^{(E)} = \mathbf{E}$, $\mathbf{T}^{(E)}$, $\mathbf{E}_i^{(E)}$, $l^{(E)}$, $F^{(E)}$, $\xi^{(E)}$, $k^{(E)}$, appropriate to the description of the equilibrium curve.

For a brief sketch of the derivation of the equations (19)–(31), we note that $F^{(E)}(t) = 0$ and $\dot{F}^{(E)}(t) = 0$ must be satisfied along the equilibrium trajectory. In addition, it is verified below [see eqs (54)–(58)] that among the constitutive equations (1)–(9) only the evolution equation for $\mathbf{E}_i$ is affected by accelerating or retarding a given strain history. That means that the map of the associated equilibrium curve is indeed defined by the system of constitutive equations included in (1)–(9), apart from the absolute value of $\dot{\mathbf{E}}_i^{(E)}$, which, like plasticity models, must be computed from the condition $\dot{F}^{(E)}(t) = 0$.

In eqns (19)–(31), the absolute value of $\dot{l}^{(E)}$ is equal to the deviatoric norm of $\dot{\mathbf{E}}_i^{(E)}$. However, we note that $\dot{l}^{(E)}$ itself does not need to be positive: although $\Theta^{(E)} > 0$, in view of eqns (31) and (12), (13), the product $\mathbf{R}_i^{(E)} \cdot \dot{\mathbf{E}}$ may be negative, in contrast to models of plasticity. Even if this inner product is negative, viscoplastic flow takes place, provided a positive overstress F exists. Thus, the governing strain-stress relation for the equilibrium curve has in general the form of a rate-dependent functional.

Let $t \in [0, T]$, and assume that inelastic loading is involved [cf. eqn (17)], as well as that

$$\mathbf{R}_i^{(E)}(t) \cdot \dot{\mathbf{E}}(t) \geqslant 0 \tag{32}$$

is satisfied. Then

$$\dot{l}^{(E)} = \|\dot{\mathbf{E}}_i^{(E)}\|_D =: \dot{s}^{(E)} \geqslant 0, \tag{33}$$

if $t \in [0, T]$, and the equations for $F \geqslant 0$ in eqns (19)–(31) represent a plasticity law because of the fact that the condition

$$F^{(E)} = 0 \quad \text{and} \quad \mathbf{R}_i^{(E)} \cdot \dot{\mathbf{E}} \geqslant 0 \tag{34a, b}$$

for plastic loading is satisfied. In (34b), equality corresponds to the case of so-called neutral loading. Now consider such a history of strain, which satisfies conditions like (17) and (34) in time intervals where $F < 0$ is not valid. Then, the constitutive equations (19)–(31) for this history of strain represent a plasticity law.

4. LIMIT PROCESSES

For convenience, we discuss limits in the constitutive model (1)–(9) with reference to strain-controlled processes. Let $\mathbf{E}(\cdot)$ be a strain history. For $t \in [0, \infty)$ and $\alpha \in (0, \infty)$ we assign to $\mathbf{E}(t)$ an accelerated ($\alpha > 1$) or retarded ($\alpha < 1$) strain curve $\boldsymbol{\mathcal{E}}^{\alpha}(\cdot)$ defined by

$$\boldsymbol{\mathcal{E}}^{\alpha}(t) := \mathbf{E}(\alpha t) = \mathbf{E}(\tau), \tag{35}$$

$$\tau := \alpha t. \tag{36}$$

The physical meaning of $\boldsymbol{\mathcal{E}}^{\alpha}(\cdot)$ is that the same strain path is traversed faster ($\alpha > 1$) or slower ($\alpha < 1$) for $t \geqslant 0$. For our purpose, it suffices to restrict the analysis in the following to the interval $[0, T]$, $0 < T < \infty$. To avoid technical details, we suppose $\mathbf{E}(\cdot)$ to have on $[0, T]$ continuous time derivatives up to second order.

The strain curve $\boldsymbol{\mathcal{E}}^{\alpha}(\cdot)$ induces a strain-controlled process in which the variables $\mathbf{T}$, $\boldsymbol{\xi}$, k, $\mathbf{E}_e$, $\mathbf{E}_i$, s, F, f, $\mathbf{R}_i$, Θ are written as Σ^{α}, Ξ^{α}, κ^{α}, $\boldsymbol{\mathcal{E}}_e^{\alpha}$, $\boldsymbol{\mathcal{E}}_i^{\alpha}$, ς^{α}, $\mathscr{F}^{\alpha}$, f^{α}, $\mathscr{R}_i^{\alpha}$, θ^{α}, respectively. According to eqns (1)–(9), we have

$$\frac{d}{dt}\boldsymbol{\mathcal{E}}^{\alpha}(t) = \frac{d}{dt}\boldsymbol{\mathcal{E}}_e^{\alpha}(t) + \frac{d}{dt}\boldsymbol{\mathcal{E}}_i^{\alpha}(t),$$

$$\mathscr{F}^{\alpha}(t) = \bar{F}(\Sigma^{\alpha}(t), \Xi^{\alpha}(t), \kappa^{\alpha}(t)) = f^{\alpha}(t) - \kappa^{\alpha}(t),$$

$$f^{\alpha}(t) := \bar{f}(\Sigma^{\alpha}(t), \Xi^{\alpha}(t)) = \sqrt{\tfrac{3}{2}(\Sigma^{\alpha}(t) - \Xi^{\alpha}(t))^{D} \cdot (\Sigma^{\alpha}(t) - \Xi^{\alpha}(t))^{D}}$$

$$\mathscr{R}_i^{\alpha}(t) := \frac{\partial \bar{F}(\Sigma^{\alpha}(t), \Xi^{\alpha}(t), \kappa^{\alpha}(t))}{\partial \Sigma^{\alpha}(t)} = \frac{3}{2f^{\alpha}(t)}(\Sigma^{\alpha}(t) - \Xi^{\alpha}(t))^{D},$$

$$\theta^{\alpha}(t) := \bar{\Theta}(\Sigma^{\alpha}(t), \Xi^{\alpha}(t), \kappa^{\alpha}(t)) \tag{37}$$

and, therefore,

$$\frac{d}{dt}\Sigma^{\alpha}(t) = 2\mu\left(\frac{d}{dt}\boldsymbol{\mathcal{E}}^{\alpha}(t) - \frac{d}{dt}\boldsymbol{\mathcal{E}}_i^{\alpha}(t)\right) + \lambda\left(\operatorname{tr}\frac{d}{dt}\boldsymbol{\mathcal{E}}^{\alpha}(t)\right)\mathbf{1}, \tag{38}$$

$$\frac{d}{dt}\boldsymbol{\mathcal{E}}_i^{\alpha}(t) = \frac{\langle \mathscr{F}^{\alpha}(t) \rangle}{r}\mathscr{R}_i^{\alpha}(t), \tag{39}$$

$$\frac{d}{dt}\Xi^{\alpha}(t) = c\frac{d}{dt}\boldsymbol{\mathcal{E}}_i^{\alpha}(t) - b\left(\frac{d}{dt}\varsigma^{\alpha}(t)\right)\Xi^{\alpha}(t), \tag{40}$$

$$\frac{d}{dt}\kappa^{\alpha}(t) = \beta(\gamma - \kappa^{\alpha}(t))\frac{d}{dt}\varsigma^{\alpha}(t), \tag{41}$$

$$\frac{d}{dt}\varsigma^{\alpha}(t) = \left\|\frac{d}{dt}\boldsymbol{\mathcal{E}}_i^{\alpha}(t)\right\|_{D} = \frac{\langle \mathscr{F}^{\alpha}(t) \rangle}{r}. \tag{42a, b}$$

In order to refer the strain-controlled process to the strain $\mathbf{E}(t)$, we introduce the following definitions, representing a rescaling of the strain-controlled process:

$$\mathbf{T}^{\alpha}(\tau) := \Sigma^{\alpha}(t), \tag{43}$$

$$\xi^{\alpha}(\tau) := \Xi^{\alpha}(t), \tag{44}$$

$$k^{\alpha}(\tau) := \kappa^{\alpha}(t), \tag{45}$$

$$\mathbf{E}_{e}^{\alpha}(\tau) := \mathcal{E}_{e}^{\alpha}(t), \tag{46}$$

$$\mathbf{E}_{i}^{\alpha}(\tau) := \mathcal{E}_{i}^{\alpha}(t), \tag{47}$$

$$F^{\alpha}(\tau) := \mathscr{F}^{\alpha}(t) = \bar{F}(\mathbf{T}^{\alpha}(\tau), \xi^{\alpha}(\tau), k^{\alpha}(\tau)) = f^{\alpha}(\tau) - k^{\alpha}(\tau), \tag{48a–d}$$

$$f^{\alpha}(\tau) := \ell^{\alpha}(t) = \bar{f}(\mathbf{T}^{\alpha}(\tau), \xi^{\alpha}(\tau)) = \sqrt{\tfrac{3}{2}(\mathbf{T}^{\alpha}(\tau) - \xi^{\alpha}(\tau))^{\mathrm{D}} \cdot (\mathbf{T}^{\alpha}(\tau) - \xi^{\alpha}(\tau))^{\mathrm{D}}}, \tag{49a–d}$$

$$\mathbf{R}_{i}^{\alpha}(\tau) := \mathscr{R}_{i}^{\alpha}(t) = \frac{\partial \bar{F}(\mathbf{T}^{\alpha}(\tau), \xi^{\alpha}(\tau), k^{\alpha}(\tau))}{\partial \mathbf{T}^{\alpha}(\tau)} = \frac{3}{2f^{\alpha}(\tau)}(\mathbf{T}^{\alpha}(\tau) - \xi^{\alpha}(\tau))^{\mathrm{D}}, \tag{50}$$

$$\Theta^{\alpha}(\tau) := \theta^{\alpha}(t) = \bar{\Theta}(\mathbf{T}^{\alpha}(\tau), \xi^{\alpha}(\tau), k^{\alpha}(\tau)). \tag{51}$$

In addition, we define

$$\frac{\mathrm{d}}{\mathrm{d}t} s^{\alpha}(t) := \left\| \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{E}_{i}^{\alpha}(t) \right\|_{\mathrm{D}} \tag{52}$$

so that, in view of eqns (42a) and (47),

$$\frac{\mathrm{d}}{\mathrm{d}\tau} s^{\alpha}(\tau) = \frac{1}{\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \varsigma^{\alpha}(t). \tag{53}$$

In the following, we focus on the nontrivial case in which the restriction of $\mathbf{E}(\cdot)$ on $[0, T]$ (i.e. the strain curve $\mathbf{E}: [0, T] \rightarrow \mathscr{L}$) induces inelastic loading for every process with $\alpha \in (0, \infty)$. Then, using the relations (35), (36) as well as (43)–(53), it is straightforward to show that eq (37) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_{e}^{\alpha}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_{i}^{\alpha}(t),$$

and eqns (38)–(42) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}^{\alpha}(t) = 2\mu\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_{i}^{\alpha}(t)\right) + \lambda\left(\operatorname{tr}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(t)\right)\mathbf{1}, \tag{54}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_{i}^{\alpha}(t) = \frac{F^{\alpha}(t)}{\alpha r}\mathbf{R}_{i}^{\alpha}(t), \tag{55}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi^{\alpha}(t) = c\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_{i}^{\alpha}(t) - b\left(\frac{\mathrm{d}}{\mathrm{d}t}s^{\alpha}(t)\right)\xi^{\alpha}(t), \tag{56}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}k^{\alpha}(t) = \beta(\gamma - k^{\alpha}(t))\frac{\mathrm{d}}{\mathrm{d}t}s^{\alpha}(t). \tag{57}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} s^{\alpha}(t) = \frac{F^{\alpha}(t)}{\alpha r} \tag{58}$$

For the five ordinary differential equations (54)–(58), the initial conditions are given by

$$\mathbf{T}^{\alpha}(t = 0) = \mathbf{T}(0), \tag{59}$$

$$\mathbf{E}_{\mathrm{i}}^{\alpha}(t = 0) = \mathbf{E}_{\mathrm{i}}(0) = \mathbf{E}^{\mathrm{D}}(t = 0) - \frac{1}{2\mu}\mathbf{T}^{\mathrm{D}}(0), \tag{60}$$

$$\xi^{\alpha}(t = 0) = \xi(0), \tag{61}$$

$$k^{\alpha}(t = 0) = k(0), \tag{62}$$

$$s^{\alpha}(t = 0) := s(0), \tag{63}$$

for every $\alpha \in (0, \infty)$. With the aid of these initial conditions, equations (54)–(58) can be integrated to obtain

$$\mathbf{T}^{\alpha}(t) = 2\mu(\mathbf{E}(t) - \mathbf{E}_{\mathrm{i}}^{\alpha}(t)) + \lambda(\operatorname{tr}\mathbf{E}(t))\mathbf{1}, \tag{64}$$

$$\mathbf{E}_{\mathrm{i}}^{\alpha}(t) = \mathbf{E}_{\mathrm{i}}(0) + \int_0^t \frac{F^{\alpha}(\bar{t})}{\alpha r}\mathbf{R}_{\mathrm{i}}^{\alpha}(\bar{t})\,\mathrm{d}\bar{t}, \tag{65}$$

$$\xi^{\alpha}(t) = \xi(0)(\mathrm{e}^{-b(s^{\alpha}(t) - s(0))}) + \int_0^t c(\mathrm{e}^{-b(s^{\alpha}(t) - s^{\alpha}(\bar{t}))})\dot{\mathbf{E}}_{\mathrm{i}}^{\alpha}(\bar{t})\,\mathrm{d}\bar{t}, \tag{66}$$

$$k^{\alpha}(t) = k(0)(\mathrm{e}^{-\beta(s^{\alpha}(t) - s(0))}) + \gamma(1 - (\mathrm{e}^{-\beta(s^{\alpha}(t) - s(0))})), \tag{67}$$

$$s^{\alpha}(t) = s(0) + \int_0^t \frac{F^{\alpha}(\bar{t})}{\alpha r}\,\mathrm{d}\bar{t}. \tag{68}$$

4.1. *An estimate for* $F^{\alpha}(t)$

Before proceeding to calculate, e.g. limits for $\alpha \to \infty$ or $\alpha \to 0$, we derive an estimate for $F^{\alpha}(t)$. In view of eqn (18), $\mathscr{F}^{\alpha}(t)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}^{\alpha}(t) + \frac{\theta^{\alpha}(t)}{r}\mathscr{F}^{\alpha}(t) = 2\mu\left(\mathscr{R}_{\mathrm{i}}^{\alpha}(t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\mathcal{E}}^{\alpha}(t)\right),$$

$$\mathscr{F}^{\alpha}(t = 0) = F(0) = \bar{F}(\mathbf{T}(0), \xi(0), k(0)). \tag{69}$$

Taking into account the relations (35)–(36) and (43)–(51), as well as the initial conditions (59)–(63), it is straightforward to conclude from eqn (69) that

$$\dot{F}^{\alpha}(t) + \frac{\Theta^{\alpha}(t)}{\alpha r}F^{\alpha}(t) = 2\mu(\mathbf{R}_{\mathrm{i}}^{\alpha}(t)\cdot\dot{\mathbf{E}}(t)),$$

$$F^{\alpha}(t = 0) = F(0), \tag{70}$$

or, after integration,

$$F^{\alpha}(t) = F(0)\left(e^{-\frac{1}{\alpha r}\int_0^t \Theta^{\alpha}(u)\,du}\right) + \int_0^t 2\mu\left(e^{-\frac{1}{\alpha r}\int_{\bar{t}}^t \Theta^{\alpha}(u)\,du}\right)(\mathbf{R}_i^{\alpha}(\bar{t})\cdot\dot{\mathbf{E}}(\bar{t}))\,d\bar{t}. \tag{71}$$

Note that, by virtue of eqns (49) and (50),

$$\|\mathbf{R}_i^{\alpha}(t)\| = \sqrt{\tfrac{3}{2}}, \tag{72}$$

and, therefore,

$$\mathbf{R}_i^{\alpha}(t)\cdot\dot{\mathbf{E}}(t) \leqslant \sqrt{\tfrac{3}{2}}\,\|\dot{\mathbf{E}}(t)\|. \tag{73}$$

Also, by eqns (51) and (12), (13),

$$\Theta^{\alpha}(t) > 3\mu > 0. \tag{74}$$

The relations (73) and (74) may be utilized to estimate the second term on the right-hand-side of eqn (71) as follows:

$$\int_0^t 2\mu\left(e^{-\frac{1}{\alpha r}\int_{\bar{t}}^t \Theta^{\alpha}(u)\,du}\right)(\mathbf{R}_i^{\alpha}(\bar{t})\cdot\dot{\mathbf{E}}(\bar{t}))\,d\bar{t} \leqslant \int_0^t 2\mu\sqrt{\frac{3}{2}}\left(e^{-\frac{3\mu}{\alpha r}(t-\bar{t})}\right)\|\dot{\mathbf{E}}(\bar{t})\|\,d\bar{t}. \tag{75}$$

Since inelastic loading is supposed for every $\alpha\in(0,\infty)$ and $t\in[0,T]$, it follows from eqns (71) and (75) that

$$0 \leqslant F^{\alpha}(t) \leqslant F(0)\left(e^{-\frac{3\mu}{\alpha r}t}\right) + \int_0^t 2\mu\sqrt{\frac{3}{2}}\left(e^{-\frac{3\mu}{\alpha r}(t-\bar{t})}\right)\|\dot{\mathbf{E}}(\bar{t})\|\,d\bar{t}, \tag{76}$$

for every $t\in[0,T]$, where again use has been made of eqn (74).

4.2. *Limits for $\alpha\to\infty$*

In the case of very fast motions ($\alpha\to\infty$), the material response reduces to the spontaneous elasticity law:

$$\mathbf{T}^{\infty}(t) := \lim_{\alpha\to\infty}\mathbf{T}^{\alpha}(t) = 2\mu(\mathbf{E}(t)-\mathbf{E}_i(0)) + \lambda(\operatorname{tr}\mathbf{E}(t))\mathbf{1}. \tag{77}$$

In addition, we have

$$\frac{d}{dt}\mathbf{T}^{\infty}(t) = 2\mu\frac{d}{dt}\mathbf{E}(t) + \lambda\left(\operatorname{tr}\frac{d}{dt}\mathbf{E}(t)\right)\mathbf{1} = \lim_{\alpha\to\infty}\frac{d}{dt}\mathbf{T}^{\alpha}(t). \tag{78}$$

To show (77), we divide (76) by αr, so that

$$0 \leqslant \frac{F^{\alpha}(t)}{\alpha r} \leqslant \frac{F(0)}{\alpha r}\left(e^{-\frac{3\mu}{\alpha r}t}\right) + \int_0^t \frac{2\mu}{\alpha r}\sqrt{\frac{3}{2}}\left(e^{-\frac{3\mu}{\alpha r}(t-\bar{t})}\right)\|\dot{\mathbf{E}}(\bar{t})\|\,d\bar{t}. \tag{79a, b}$$

Taking into account the properties of $\mathbf{E}(\cdot)$, applying the theorem on uniform convergence and Riemann integration [see e.g. Apostol (1965: p. 399)], it can be readily shown that the right-hand-side of eqn (79) converges to 0 as $\alpha\to\infty$. Consequently, (79) implies

$$\lim_{\alpha \to \infty} \frac{F^{\alpha}(t)}{\alpha r} = 0. \tag{80}$$

By following similar steps to those in deriving eqn (80), the results

$$\lim_{\alpha \to \infty} (\mathbf{E}_i^{\alpha}(t), s^{\alpha}(t), \xi^{\alpha}(t), k^{\alpha}(t)) = (\mathbf{E}_i(0), s(0), \xi(0), k(0)) \tag{81}$$

can be deduced from eqns (65)–(68). In other words, the internal variables are frozen at their initial value. Finally, by substituting eqn (81) into (64), one obtains (77).
The proof of eqn (78) is based on the result

$$\lim_{\alpha \to \infty} \dot{\mathbf{E}}_i^{\alpha}(t) = \mathbf{0}, \tag{82}$$

which follows from eqns (80), (55) and the fact that $\lim_{\alpha \to \infty} \|\mathbf{R}_i^{\alpha}(t)\| = \sqrt{\frac{3}{2}}$ [see eqn (72)]. Then, comparing eqn (77) with (54) together with (82), one obtains (78).

4.3 *Limits for* $\alpha \to 0$

4.3.1. *Calculation of* $\lim_{\alpha \to 0} F^{\alpha}(t)$. Let $t \in (0, T]$. In view of the fact that

$$\lim_{\alpha \to 0} \left(e^{-\frac{3\mu}{\alpha r}(t-\bar{t})} \right) \|\dot{\mathbf{E}}(\bar{t})\| = \begin{cases} 0 & \text{if} \quad \bar{t} \in [0, t) \\ \|\dot{\mathbf{E}}(t)\| < \infty & \text{if} \quad \bar{t} = t \end{cases},$$

($\bar{t} \in [0, T]$), and using Arzelà's theorem on bounded convergence and Riemann integration [see e.g. Apostol (1965: p.405)], one finds that

$$\lim_{\alpha \to 0} \int_0^t \left(e^{-\frac{3\mu}{\alpha r}(t-\bar{t})} \right) \|\dot{\mathbf{E}}(\bar{t})\| \, d\bar{t} = 0.$$

Then, from eqn (76) it follows that

$$\lim_{\alpha \to 0} F^{\alpha}(t) = \begin{cases} F(0) & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \in (0, \infty] \end{cases}. \tag{83a, b}$$

4.3.2. *Calculation of* $\lim_{\alpha \to 0} (1/\alpha r) F^{\alpha}(t)$. The calculation of the limit $(1/\alpha r) F^{\alpha}(t)$ as $\alpha \to 0$ is based on the following theorem.

Theorem: Let $\alpha \in D := (0, 1]$, and let the real-valued functions $\varphi_{\alpha}(t)$ and $g_{\alpha}(t)$ be smooth for $t \in [0, T], 0 < T < \infty$. Furthermore, suppose that the functions $\varphi_{\alpha}(t)$ are uniformly bounded from below in the form

$$\varphi_{\alpha}(t) \geq M > 0, \quad M = \text{const.}, \tag{84}$$

and that $\lim_{\alpha \to 0} g_{\alpha}(t)$, as well as $\lim_{\alpha \to 0} \dot{g}_{\alpha}(t)$ and $\lim_{\alpha \to 0} \dot{\varphi}_{\alpha}(t)$, are bounded. Then

$$\lim_{\alpha \to 0} \int_0^t \frac{A}{\alpha} \left(e^{-\frac{A}{\alpha} \int_{\bar{t}}^{t} \varphi_{\alpha}(u) du} \right) g_{\alpha}(\bar{t}) \, d\bar{t} = \lim_{\alpha \to 0} \frac{g_{\alpha}(t)}{\varphi_{\alpha}(t)}, \tag{85}$$

where $A = \text{const.} > 0$.

Proof: The assumed properties for $g_{\alpha}(t)$ and $\varphi_{\alpha}(t)$ allow one to establish the result

$$\int_0^t \frac{A}{\alpha}\left(e^{-\frac{A}{\alpha}\int_{\bar{t}}^t \varphi_\alpha(u)\,du}\right) g_\alpha(\bar{t})\,d\bar{t} = \int_0^t \frac{d}{d\bar{t}}\left(e^{-\frac{A}{\alpha}\int_{\bar{t}}^t \varphi_\alpha(u)\,du}\right)\frac{g_\alpha(\bar{t})}{\varphi_\alpha(\bar{t})}\,d\bar{t}$$

$$= \frac{g_\alpha(t)}{\varphi_\alpha(t)} - \left(e^{-\frac{A}{\alpha}\int_0^t \varphi_\alpha(u)\,du}\right)\frac{g_\alpha(0)}{\varphi_\alpha(0)} - \int_0^t \left(e^{-\frac{A}{\alpha}\int_{\bar{t}}^t \varphi_\alpha(u)\,du}\right)\left(\frac{g_\alpha(\bar{t})}{\varphi_\alpha(\bar{t})}\right)^{\bullet} d\bar{t}, \quad (86)$$

where use has been made of the formula for integration by parts. Then, eqn (85) may be obtained from eqn (86), if in addition the relations

$$\lim_{\alpha\to 0}\left(e^{-\frac{A}{\alpha}\int_0^t \varphi_\alpha(u)\,du}\right)\frac{g_\alpha(0)}{\varphi_\alpha(0)} = 0 \quad (87)$$

and

$$\lim_{\alpha\to 0}\int_0^t \left(e^{-\frac{A}{\alpha}\int_{\bar{t}}^t \varphi_\alpha(u)\,du}\right)\left(\frac{g_\alpha(\bar{t})}{\varphi_\alpha(\bar{t})}\right)^{\bullet} d\bar{t} = 0 \quad (88)$$

are satisfied. Equation (87) follows from the inequalities

$$0 \leqslant e^{-\frac{A}{\alpha}\int_0^t \varphi_\alpha(u)\,du} \leqslant e^{-\frac{AM}{\alpha}t},$$

which are implied by eqn (84), as well as from the fact that $g_\alpha(t)$ and $\varphi_\alpha(t)$ are bounded. To obtain eqn (88), we recall from the properties of $g_\alpha(t)$ and $\varphi_\alpha(t)$ that $(g_\alpha(t)/\varphi_\alpha(t))^{\bullet}$ is bounded for every $\alpha \in D$, and in particular for the limit as $\alpha \to 0$. Thus, there exists a constant L, $0 \leqslant L < \infty$, so that the relation

$$\sup_{\bar{t}\in[0,t]}\left|\left(\frac{g_\alpha(\bar{t})}{\varphi_\alpha(\bar{t})}\right)^{\bullet}\right| \leqslant L$$

is satisfied for every $\alpha \in D$. Then,

$$0 \leqslant \left|\int_0^t \left(e^{-\frac{A}{\alpha}\int_{\bar{t}}^t \varphi_\alpha(u)\,du}\right)\left(\frac{g_\alpha(\bar{t})}{\varphi_\alpha(\bar{t})}\right)^{\bullet} d\bar{t}\right|$$

$$\leqslant \int_0^t \left(e^{-\frac{AM}{\alpha}(t-\bar{t})}\right)\left|\left(\frac{g_\alpha(\bar{t})}{\varphi_\alpha(\bar{t})}\right)^{\bullet}\right| d\bar{t}$$

$$\leqslant L\int_0^t \left(e^{-\frac{AM}{\alpha}(t-\bar{t})}\right) d\bar{t}$$

$$= L\frac{\alpha}{AM}\left(1 - e^{-\frac{AM}{\alpha}(t)}\right).$$

Therefore,

$$\lim_{\alpha\to 0}\left|\int_0^t \left(e^{-\frac{A}{\alpha}\int_{\bar{t}}^t \varphi_\alpha(u)\,du}\right)\left(\frac{g_\alpha(\bar{t})}{\varphi_\alpha(\bar{t})}\right)^{\bullet} d\bar{t}\right| = 0.$$

This relation implies eqn (88), which completes the proof of the theorem.

Now, we shall use the theorem to show first that $\lim_{\alpha\to 0}(F^\alpha(t)/\alpha r)$ is bounded for $t \in (0, T]$. With respect to inequality (79), we have to show that

$$\lim_{\alpha\to 0}\int_0^t \frac{3\mu}{\alpha r}\left(e^{-\frac{3\mu}{\alpha r}(t-\bar t)}\right)\|\dot{\mathbf{E}}(\bar t)\|_D\,d\bar t$$

is bounded. By setting $A = 3\mu/r$, $\varphi_\alpha(t) = 1$ and $g_\alpha(t) = \|\dot{\mathbf{E}}(t)\|_D$ in eqn (85),

$$\lim_{\alpha\to 0}\int_0^t \frac{3\mu}{\alpha r}\left(e^{-\frac{3\mu}{\alpha r}(t-\bar t)}\right)\|\dot{\mathbf{E}}(\bar t)\|_D\,d\bar t = \|\dot{\mathbf{E}}(t)\|_D < \infty.$$

Because of the properties of the strain curve considered, one can establish, using this relation in inequality (79), the desired estimate

$$\lim_{\alpha\to 0}\frac{F^\alpha(t)}{\alpha r} \leqslant \|\dot{\mathbf{E}}(t)\|_D < \infty \quad \text{for } t\in(0,T]. \tag{89}$$

Next, return to eqn (71), and divide both sides by αr:

$$\frac{1}{\alpha r}F^\alpha(t) = \frac{F(0)}{\alpha r}\left(e^{-\frac{1}{\alpha r}\int_0^t\Theta^\alpha(u)\,du}\right) + \int_0^t \frac{2\mu}{\alpha r}\left(e^{-\frac{1}{\alpha r}\int_{\bar t}^t\Theta^\alpha(u)\,du}\right)\mathbf{R}_1^\alpha(\bar t)\cdot\dot{\mathbf{E}}(\bar t)\,d\bar t. \tag{90}$$

In this equation, we form the limit for $\alpha\to 0$ and $t\in(0,T]$. Evidently,

$$\lim_{\alpha\to 0}\frac{1}{\alpha r}F^\alpha(t) = \lim_{\alpha\to 0}\int_0^t \frac{2\mu}{\alpha r}\left(e^{-\frac{1}{\alpha r}\int_{\bar t}^t\Theta^\alpha(u)\,du}\right)\mathbf{R}_1^\alpha(\bar t)\cdot\dot{\mathbf{E}}(\bar t)\,d\bar t \tag{91}$$

for $t\in(0,T]$.

By using the theorem above, with $A = 1/r$ and

$$\varphi_\alpha(t) = \Theta^\alpha(t), \tag{92}$$

$$g_\alpha(t) = \mathbf{R}_1^\alpha(t)\cdot\dot{\mathbf{E}}(t), \tag{93}$$

eqn (91) leads to

$$\lim_{\alpha\to 0}\frac{1}{\alpha r}F^\alpha(t) = \lim_{\alpha\to 0}\frac{2\mu\mathbf{R}_1^\alpha(t)\cdot\dot{\mathbf{E}}(t)}{\Theta^\alpha(t)}, \quad \text{for } t\in(0,\infty]. \tag{94}$$

Of course, to justify the application of the theorem, we have to show that the conditions for (85) are satisfied. In essence, we must show that $g_\alpha(t)$, defined by eqn (93), as well as $\dot g_\alpha(t)$, are bounded for $\alpha\to 0$. Also, we must show for $\varphi_\alpha(t)$, defined by eqn (92), that $\lim_{\alpha\to 0}\dot\varphi_\alpha(t)$ is bounded [$\varphi_\alpha(t)$ satisfies the condition (74)].

Since $|g_\alpha(t)| \leqslant \sqrt{\tfrac{3}{2}}\|\dot{\mathbf{E}}(t)\|$ by eqn (72), we have $|\lim_{\alpha\to 0} g_\alpha(t)| < \infty$. In order to prove that $\lim_{\alpha\to 0}\dot g_\alpha(t)$ is also bounded, it suffices to verify that $\|\dot{\mathbf{R}}_1^\alpha(t)\|$ is bounded for $\alpha\to 0$. In view of (50)$_1$,

$$\dot{\mathbf{R}}_1^\alpha(t) = \frac{3}{2f^\alpha(t)}(\dot{\mathbf{T}}^\alpha(t)-\dot{\xi}^\alpha(t))^D - \frac{9(\mathbf{T}^\alpha(t)-\xi^\alpha(t))^D\cdot(\dot{\mathbf{T}}^\alpha(t)-\dot{\xi}^\alpha(t))}{4(f^\alpha(t))^3}(\mathbf{T}^\alpha(t)-\xi^\alpha(t))^D. \tag{95}$$

It is remarked that the back stress and the variable for the isotropic hardening satisfy the inequalities (14) and (15) for every process and, therefore, $\lim_{\alpha\to 0}\xi^\alpha(t)$ and $\lim_{\alpha\to 0}k^\alpha(t)$ are bounded. Thus, with respect to eqn (95), $\dot{\mathbf{R}}_i^\alpha(t)$ is bounded for $\alpha \to 0$ if $(\mathbf{T}^\alpha(t))^D$, $\dot{\mathbf{T}}^\alpha(t)$ and $\dot{\xi}^\alpha(t)$ are bounded for $\alpha \to 0$ ($f^\alpha(t) \geqslant k^\alpha(t) \geqslant k_0 > 0$ for viscoplastic loading). Clearly, by the definitions (48c) and (49c), as well as the result (83b), $\lim_{\alpha \to 0} f^\alpha(t)$ and, therefore, $\lim_{\alpha \to 0}(\mathbf{T}^\alpha(t))^D$, is bounded for $t \in (0, T]$.

Regarding eqns (55), (56) and (58), $\dot{\xi}^\alpha(t)$ is proportional to $\mathbf{R}_i^\alpha(t)$ and $\xi^\alpha(t)$ (which remain bounded for $\alpha \to 0$). In addition, $\dot{\xi}^\alpha(t)$ is proportional to $(1/\alpha r)F^\alpha(t)$, which, by eqn (89), remains bounded as well for $\alpha \to 0$ and $t \in (0, T]$. Thus, $\|\lim_{\alpha \to 0} \dot{\xi}^\alpha(t)\| < \infty$ for $t \in (0, T]$. By means of the same arguments, and with respect to eqn (54), it is also possible to show that $\|\lim_{\alpha \to 0} \dot{\mathbf{T}}^\alpha(t)\| < \infty$ for $t \in (0, T]$.

Furthermore, following similar steps, it can be shown that $\|\lim_{\alpha \to 0} \dot{\Theta}^\alpha(t)\| < \infty$ for $t \in (0, T]$, where $\Theta^\alpha(t)$ is defined by eqns (51) and (12). This completes the verification of eqn (94). So, taking into account eqn (94), it follows from (90) that

$$\lim_{\alpha \to 0} \frac{F^\alpha(t)}{\alpha r} = \begin{cases} 0 & \text{if} \quad t = 0 \quad \text{and} \quad F(0) = 0 \\ \infty & \text{if} \quad t = 0 \quad \text{and} \quad F(0) > 0 \\ 2\mu \lim_{\alpha \to 0} \dfrac{\mathbf{R}_i^\alpha(t) \cdot \dot{\mathbf{E}}(t)}{\Theta^\alpha(t)} & \text{if} \quad t \in (0, T] \end{cases} . \tag{96a–c}$$

4.3.3. *Calculation of the limit functions.* Let $\mathscr{M}^\alpha(t)$ be defined by

$$\mathscr{M}^\alpha(t) := (\mathbf{T}^\alpha(t), \mathbf{E}_i^\alpha(t), \xi^\alpha(t), k^\alpha(t), s^\alpha(t)).$$

We shall calculate the limit functions $\lim_{\alpha \to 0} \mathscr{M}^\alpha(t)$ implicitly, as solutions of corresponding ordinary differential equations. For this purpose, we distinguish between two cases, $t \in (0, T]$ or $t = 0$.

(1) $t \in (0, T]$. By eqn (83b), the overstress $F^\alpha(t)$ vanishes as $\alpha \to 0$ for every $t \in (0, T]$. This means that, as $\alpha \to 0$, the state trajectory converges to the associated equilibrium trajectory determined by the given strain curve $\mathbf{E}: [0, T] \to \mathscr{L}$ and the assumed initial conditions (59)–(63). Thus

$$\lim_{\alpha \to 0} \mathscr{M}^\alpha(t) = \mathscr{M}^{(E)}(t), \tag{97}$$

where

$$\mathscr{M}^{(E)}(t) := (\mathbf{T}^{(E)}(t), \mathbf{E}_i^{(E)}(t), \xi^{(E)}(t), k^{(E)}(t), s^{(E)}(t)).$$

By virtue of eqns (79a) and (94), as well as of the fact that $\lim_{\alpha \to 0} \Theta^\alpha(t) > 3\mu > 0$, viscoplastic loading for each $\alpha \in (0, 1]$ implies

$$\lim_{\alpha \to 0} \mathbf{R}_i^\alpha(t) \cdot \dot{\mathbf{E}}(t) \geqslant 0. \tag{98}$$

Equality in (98) corresponds to the case of so-called neutral loading. In other words, for this case, the strain-stress relation of the associated equilibrium curve reduces to the plasticity law defined by the differential equations (23)–(34). In addition, using the definition

$$\dot{\mathscr{M}}^{(E)}(t) := (\dot{\mathbf{T}}^{(E)}(t), \dot{\mathbf{E}}_i^{(E)}(t), \dot{\xi}^{(E)}(t), \dot{k}^{(E)}(t), \dot{s}^{(E)}(t)), \tag{99}$$

we have

$$\frac{d}{dt}\left(\lim_{\alpha \to 0} \mathcal{M}^{\alpha}(t)\right) = \dot{\mathcal{M}}^{(E)}(t). \tag{100}$$

It is also interesting to know the behaviour for the functions $\lim_{\alpha \to 0} \dot{\mathcal{M}}^{\alpha}(t)$, where $\dot{\mathcal{M}}^{\alpha}(t)$ is defined similarly to eqn (99). To determine this, we insert the relations

$$\lim_{\alpha \to 0} \mathbf{R}_i^{\alpha}(t) = \mathbf{R}_i^{(E)},$$

$$\lim_{\alpha \to 0} \Theta^{\alpha}(t) = \Theta^{(E)},$$

in (96c), so that

$$\lim_{\alpha \to 0} \frac{F^{\alpha}(t)}{\alpha r} = \dot{s}^{(E)}$$

by virtue of eqns (30) and (33). Then it is clear that eqns (54)–(58) imply

$$\lim_{\alpha \to 0} \dot{\mathcal{M}}^{\alpha}(t) = \dot{\mathcal{M}}^{(E)}(t). \tag{101}$$

(2) $t = 0$. In this case, we have

$$\lim_{\alpha \to 0} \mathcal{M}^{\alpha}(0) = \mathcal{M}(0) := (\mathbf{T}(0), \mathbf{E}_i(0), \xi(0), k(0), s(0)). \tag{102}$$

If $\mathcal{M}(0) \neq \mathcal{M}^{(E)}(0)$, the limit functions $\lim_{\alpha \to 0} \mathcal{M}^{\alpha}(t)$ are discontinuous from the right at the point $t = 0$. If $\mathcal{M}(0) = \mathcal{M}^{(E)}(0)$, the limit functions $\lim_{\alpha \to 0} \mathcal{M}^{\alpha}(t)$ are smooth at $t = 0$. For the time derivatives of the limit functions,

$$\left[\frac{d}{dt}\left(\lim_{\alpha \to 0} \mathcal{M}^{\alpha}(t)\right)\right]_{t=0} = \begin{cases} \dot{\mathcal{M}}^{(E)}(t)|_{t=0} & \text{if} \quad \mathcal{M}(0) = \mathcal{M}^{(E)}(0) \\ \text{does not exist in the proper sense} & \text{if} \quad \mathcal{M}(0) \neq \mathcal{M}^{(E)}(0) \end{cases} \tag{103}$$

holds.

On the other hand, from eqns (96a,b) and (54)–(58), we have

$$\lim_{\alpha \to 0} [\dot{\mathcal{M}}^{\alpha}(t)]_{t=0} = \begin{cases} \mathcal{A} & \text{if} \quad \mathcal{M}(0) = \mathcal{M}^{(E)}(0) \\ \text{does not exist in the proper sense} & \text{if} \quad \mathcal{M}(0) \neq \mathcal{M}^{(E)}(0) \end{cases}, \tag{104}$$

where

$$\mathcal{A} := ([2\mu\dot{\mathbf{E}}(t) + \lambda(\mathrm{tr}\,\dot{\mathbf{E}}(t)\mathbf{1}]_{t=0}, \mathbf{0}, \mathbf{0}, 0, 0).$$

5. DISCUSSION AND CONCLUDING REMARKS

As it has been shown [see eqn (77)], the material response approaches the spontaneous elasticity law for very fast motions ($\alpha \to \infty$). In particular, as $\alpha \to \infty$, not only $\mathbf{T}^{\alpha}(t)$ converges to $\mathbf{T}^{\infty}(t)$, but also [see eqn (78)] $(d/dt)\,\mathbf{T}^{\alpha}(t)$ to $(d/dt)\,\mathbf{T}^{\infty}(t)$. Obviously, a limit process analogous to that for $\alpha \to \infty$, but depending on material parameters, can also be carried out. To this end, one has to consider $\alpha = 1$ and $r \to \infty$.

The main results for very slow motions ($\alpha \to 0$) are that

$$\lim_{\alpha\to 0}\mathscr{M}^{\alpha}(t)=\begin{cases}\mathscr{M}(0) & \text{if}\quad t=0\\ \mathscr{M}^{(E)}(t) & \text{if}\quad t\in(0,T]\end{cases}\tag{105}$$

by virtue of eqns (97) and (102),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lim_{\alpha\to 0}\mathscr{M}^{\alpha}(t)\right)=\begin{cases}\dot{\mathscr{M}}^{(E)}(t)|_{t=0} & \text{if}\quad t=0\quad\text{and}\quad \mathscr{M}(0)=\mathscr{M}^{(E)}(0)\\ \text{does not exist in the proper sense} & \text{if}\quad t=0\quad\text{and}\quad \mathscr{M}(0)\neq\mathscr{M}^{(E)}(0)\\ \dot{\mathscr{M}}^{(E)}(t) & \text{if}\quad t\in(0,T]\end{cases}\tag{106a–c}$$

by virtue of (100), (103), and

$$\lim_{\alpha\to 0}\dot{\mathscr{M}}^{\alpha}(t)=\begin{cases}\mathscr{A} & \text{if}\quad t=0\quad\text{and}\quad \mathscr{M}(0)=\mathscr{M}^{(E)}(0)\\ \text{does not exist in the proper sense} & \text{if}\quad t=0\quad\text{and}\quad \mathscr{M}(0)\neq\mathscr{M}^{(E)}(0).\\ \dot{\mathscr{M}}^{(E)}(t) & \text{if}\quad t\in(0,T]\end{cases}\tag{107a–c}$$

in view of eqns (101) and (104).

The meaning of these results is obvious: if $\mathscr{M}(0)\neq\mathscr{M}^{(E)}(0)$ and $\alpha\to 0$, the actual variables are discontinuous, according to eqn (105), at $t=0$ from the initial value $\mathscr{M}(0)$ to the value $\mathscr{M}^{(E)}(0)$ and equal $\mathscr{M}^{(E)}(t)$ for $t\in(0,T]$. If $\mathscr{M}(0)=\mathscr{M}^{(E)}(0)$, then, according to eqn (105), $\lim_{\alpha\to 0}\mathscr{M}^{\alpha}(t)=\mathscr{M}^{(E)}(t)$ everywhere on $[0,T]$. However, the behaviour for the time derivatives is different. Although $\lim_{\alpha\to 0}\dot{\mathscr{M}}^{\alpha}(t)$ converges to $\dot{\mathscr{M}}^{(E)}(t)$ for $t\in(0,T]$ as well, at $t=0$ and for $\mathscr{M}(0)=\mathscr{M}^{(E)}(0)$, we have $\lim_{\alpha\to 0}[\dot{\mathscr{M}}^{\alpha}(t)]_{t=0}\neq[\dot{\mathscr{M}}^{(E)}]_{t=0}$ by eqn (107a) [cf. also eqn (106a)].

It should be mentioned that the analysis for $\alpha\to 0$ remains the same if limit processes depending on material parameters are studied, i.e. if we consider $r\to 0$ at $\alpha=1$.

According to Tsakmakis (1994), a material is defined to be of viscoplastic type if the stress is determined by a rate-dependent functional of the strain which satisfies the following two properties: (1) the functional assigns an equilibrium curve to each process, where the equilibrium stress is given as a functional of the strain, and (2) in the case of infinitely slow motions, the constitutive relation between the strain and the stress reduces to the constitutive relation governing the response of the associated equilibrium curve. At the same time, the constitutive relations for the associated equilibrium curve reduce to a plasticity law.

Clearly, the investigations in the present work indicate that the constitutive model considered in Section 3 represents in fact a rate-dependent functional of the viscoplastic type, according to the definition above.

Finally, we want to discuss a simple modification of the constitutive model given in Section 3. It is obvious, in view of the foregoing analysis for very fast and very slow motions, that in the constitutive model (1)–(9), only the evolution equation for $\mathbf{E}_i$ can be affected by a formulation of the constitutive equations with respect to a transformed time z, where

$$\mathrm{d}z=\Phi(t)\,\mathrm{d}t.$$

In this equation, Φ is defined to be a constitutive function. For practical purposes, Φ can be chosen as a function of the overstress F:

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\hat{\Phi}(\langle F(t)\rangle).\tag{108}$$

Then, with respect to eqn (5), the modified evolution equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{E}_i = \frac{\langle F \rangle}{r}\mathbf{R} \tag{109a}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_i = \frac{\varphi(\langle F \rangle)}{r}\mathbf{R}_i, \tag{109b}$$

where $\varphi(\langle F \rangle) := \langle F \rangle \hat{\Phi}(\langle F \rangle)$. It appears convenient to require for the function $\hat{\Phi} : [0, \infty] \rightarrow R$, the properties

$$\hat{\Phi}(0) = 0 \quad \text{as well as} \quad \hat{\Phi}(x) > 0 \quad \text{and} \quad \frac{\mathrm{d}\hat{\Phi}(x)}{\mathrm{d}x} \geqslant 0 \quad \text{for} \quad x \in (0, \infty). \tag{110}$$

As a particular case, we have

$$\varphi(\langle F \rangle) = \langle F \rangle^m, \tag{111}$$

m being a material parameter. Equations (1)–(9), together with the evolution equation for $\mathbf{E}_i$ replaced by equations (109)–(111), constitute a viscoplasticity model which is the basis for many works by Chaboche [see e.g. Chaboche (1977, 1993)].

It is not difficult to see that time transformations, e.g., of the form (108) and (110), which represent a monotonous function between t and z, do not affect our results for very fast and very slow motions. We have only to regard the accelerated or retarded strain curve in our investigations as related to a given strain curve, which is parameterized by the time z: because of the monotonous relation between t and z, retarding or accelerating a strain path parameterized by z implies retarding or accelerating the same strain path parameterized by t, respectively.

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